Int. J. Solids Structures Vol. 29, No. 16, pp. 2063–2076, 1992
Printed in Great Britain.

0020-7683/92 $5.00 + .00

CONJUGATE STRESS AND TENSOR EQUATION $\sum_{r=1}^{m} \mathbf{U}^{m-r}\mathbf{X}\mathbf{U}^{r-1} = \mathbf{C}$

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(Received 18 April 1991)

Abstract—In this paper we seek an explicit tensorial expression which, for each integer $m \neq 0$, gives the stress $\mathbf{T}^{(m)}$ conjugate to the strain $\mathbf{E}^{(m)}$ in the Seth–Hill class. The preceding problem leads us to find an intrinsic (i.e. coordinate-free) representation of the solution of the tensor equation displayed in the title. We obtain the requisite representation by using Hill's principal axis method and the representation theorem of isotropic tensor functions. We illustrate the general procedure to obtain an explicit tensorial expression for $\mathbf{T}^{(m)}$ by working out the instance $m = -3$ in detail.

1. INTRODUCTION

The notion of work conjugacy of stress and strain in solid mechanics was introduced by Hill (1968) and by Macvean (1968). In the sense of conjugacy, a class of stress measures may be derived in a natural way. Some of these stress measures have already proved useful [cf. e.g. Hill (1968); Guo (1980)].

According to Hill, the definition of work conjugacy is as follows: For a given Lagrangean-type strain **E**, if there is a symmetric second-order tensor **T** such that the stress power (the rate of specific work) per unit reference volume

$$\dot{w} = \mathrm{III}\,\boldsymbol{\sigma}:\mathbf{D} \quad (\equiv \mathrm{III}\,\mathrm{tr}\,(\boldsymbol{\sigma}\mathbf{D})) \tag{1}$$

can be recast as

$$\dot{w} = \mathbf{T}:\dot{\mathbf{E}} \quad (\equiv \mathrm{tr}\,(\mathbf{T}\dot{\mathbf{E}})), \tag{2}$$

then **T** may serve as a stress measure and (**T**, **E**) is called a conjugate pair. Here $\boldsymbol{\sigma}$ is the Cauchy stress tensor, **D** is the stretching tensor (i.e. the symmetric part of the velocity gradient **L**) and $\mathrm{III} = \det \mathbf{U}$ is the third principal invariant of the right stretch tensor **U**.

Let $\mathbf{U} = \sum_i \lambda_i \mathbf{N}_i \otimes \mathbf{N}_i$, where $\{\lambda_i\}$ and $\{\mathbf{N}_i\}$ are the principal stretches and the subordinate orthonormal eigenvectors of **U**, respectively. Hill (1968) proposed a class of strain measures given by

$$\mathbf{E}(\mathbf{U}) = \sum_i f(\lambda_i)\mathbf{N}_i \otimes \mathbf{N}_i, \tag{3}$$

where $f(\cdot)$ is a smooth strictly-increasing scalar function that satisfies $f(1) = 0$ and $f'(1) = 1$. Earlier Seth (1964) discussed a subclass of strain measures **E**(**U**) indexed by the parameter m, where the function

$$f(\lambda) = \begin{cases} \dfrac{1}{m}(\lambda^m - 1), & \text{if } m \neq 0, \\ \ln \lambda, & \text{if } m = 0. \end{cases} \tag{4}$$

We suggest calling strain tensors from this subclass the Seth–Hill strain measures, and we denote them with superscript by

$$\mathbf{E}^{(m)} = \frac{1}{m}\sum_i (\lambda_i^m - 1)\mathbf{N}_i \otimes \mathbf{N}_i = \frac{1}{m}(\mathbf{U}^m - \mathbf{I}), \quad \text{if } m \neq 0,$$

$$\mathbf{E}^{(0)} = \sum_i \ln \lambda_i \mathbf{N}_i \otimes \mathbf{N}_i = \ln \mathbf{U}. \tag{5}$$

We mention the following often used strain measures as special cases of (5):

(i) Green's strain: $\mathbf{E}^{(2)} = \frac{1}{2}(\mathbf{U}^2 - \mathbf{I})$,
(ii) Almansi strain: $\mathbf{E}^{(-2)} = \frac{1}{2}(\mathbf{I} - \mathbf{U}^{-2})$,
(iii) Nominal strain: $\mathbf{E}^{(1)} = \mathbf{U} - \mathbf{I}$,
(iv) Logarithmic strain: $\mathbf{E}^{(0)} = \ln \mathbf{U}$.

The stress tensors conjugate to $\mathbf{E}^{(2)}$, $\mathbf{E}^{(-2)}$ and $\mathbf{E}^{(1)}$ are well known [cf. Hill (1978); Guo (1984)]. They are the second Piola–Kirchhoff stress tensor

$$\mathbf{T}^{(2)} = \mathrm{III}\, \mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-T}, \tag{6}$$

(where $\mathbf{F}$ is the deformation gradient), the weighted convected stress tensor

$$\mathbf{T}^{(-2)} = \mathrm{III}\, \mathbf{F}^{T}\boldsymbol{\sigma}\mathbf{F} \tag{7}$$

[Truesdell and Noll (1965) called $\mathbf{F}^T\boldsymbol{\sigma}\mathbf{F}$ the convected stress] and the Jaumann stress tensor

$$\mathbf{T}^{(1)} = \tfrac{1}{2}(\mathbf{T}^{(2)}\mathbf{U} + \mathbf{U}\mathbf{T}^{(2)}). \tag{8}$$

Recently Hoger (1987) found a tensorial expression for the stress $\mathbf{T}^{(0)}$ conjugate to $\ln \mathbf{U}$. From

$$\dot{\mathbf{E}}^{(-1)} = -\dot{\overline{\mathbf{U}^{-1}}},$$

$$\dot{\mathbf{E}}^{(-2)} = -\tfrac{1}{2}(\mathbf{U}^{-1}\dot{\overline{\mathbf{U}^{-1}}} + \dot{\overline{\mathbf{U}^{-1}}}\mathbf{U}^{-1}),$$

and

$$w = \mathbf{T}^{(-2)} : \dot{\mathbf{E}}^{(-2)} = \mathbf{T}^{(-2)} : \tfrac{1}{2}(\mathbf{U}^{-1}\dot{\mathbf{E}}^{(-1)} + \dot{\mathbf{E}}^{(-1)}\mathbf{U}^{-1})$$
$$= \tfrac{1}{2}(\mathbf{T}^{(-2)}\mathbf{U}^{-1} + \mathbf{U}^{-1}\mathbf{T}^{(-2)}) : \dot{\mathbf{E}}^{(-1)},$$

we may add the conjugate stress

$$\mathbf{T}^{(-1)} = \tfrac{1}{2}(\mathbf{T}^{(-2)}\mathbf{U}^{-1} + \mathbf{U}^{-1}\mathbf{T}^{(-2)}) \tag{9}$$

to the list of conjugate stress tensors for which tensorial expressions have been found.

For positive integers m, we obtain by differentiating (5)

$$\dot{\mathbf{E}}^{(m)} = \frac{1}{m}\sum_{r=1}^{m} \mathbf{U}^{m-r}\dot{\mathbf{U}}\mathbf{U}^{r-1}.$$

Substituting the preceding expression into the identity

$$\mathbf{T}^{(1)} : \dot{\mathbf{E}}^{(1)} = \mathbf{T}^{(m)} : \dot{\mathbf{E}}^{(m)}$$

yields

$$\mathbf{T}^{(1)} : \dot{\mathbf{U}} = \frac{1}{m}\left(\sum_{r=1}^{m} \mathbf{U}^{m-r}\mathbf{T}^{(m)}\mathbf{U}^{r-1}\right) : \dot{\mathbf{U}}.$$

The arbitrariness of $\dot{\mathbf{U}}$ implies that the stress tensor $\mathbf{T}^{(m)}$ conjugate to $\mathbf{E}^{(m)}$ satisfies the tensor equation

$$\sum_{r=1}^{m} \mathbf{U}^{m-r}\mathbf{X}\mathbf{U}^{r-1} = m\mathbf{T}^{(1)}. \tag{10}$$

Analogously, for negative integers $-m$ (where $m > 0$), we have

$$\dot{\mathbf{E}}^{(-m)} = -\frac{1}{m}\sum_{r=1}^{m} \mathbf{U}^{r-m}\dot{\overline{\mathbf{U}^{-1}}}\mathbf{U}^{1-r},$$

$$\mathbf{T}^{(-1)} : \dot{\mathbf{E}}^{(-1)} = \mathbf{T}^{(-m)} : \dot{\mathbf{E}}^{(-m)},$$

$$\mathbf{T}^{(-1)} : \dot{\overline{\mathbf{U}^{-1}}} = \frac{1}{m}\left(\sum_{r=1}^{m} \mathbf{U}^{r-m}\mathbf{T}^{(-m)}\mathbf{U}^{1-r}\right) : \dot{\overline{\mathbf{U}^{-1}}},$$

and we draw the analogous conclusion that the stress tensor $\mathbf{T}^{(-m)}$ conjugate to $\mathbf{E}^{(-m)}$ satisfies the tensor equation

$$\sum_{r=1}^{m} \mathbf{U}^{r-m}\mathbf{X}\mathbf{U}^{1-r} = m\mathbf{T}^{(-1)}. \tag{11}$$

Pre- and postmultiplying (11) by $\mathbf{U}^{m-1}$, we get an equivalent equation

$$\sum_{r=1}^{m} \mathbf{U}^{m-r}\mathbf{X}\mathbf{U}^{r-1} = m\mathbf{U}^{m-1}\mathbf{T}^{(-1)}\mathbf{U}^{m-1}, \tag{12}$$

which differs from (10) only by the right-hand side. Thus, the crucial point in obtaining an explicit formula for the conjugate stress $\mathbf{T}^{(m)}$ or $\mathbf{T}^{(-m)}$, where m is a positive integer, is to solve the tensor equation

$$\sum_{r=1}^{m} \mathbf{U}^{m-r}\mathbf{X}\mathbf{U}^{r-1} = \mathbf{C}, \tag{13}$$

where $\mathbf{C}$ is a given symmetric tensor. We shall investigate eqn (13) in the next section.

2. TENSOR EQUATION $\sum_{r=1}^{m} \mathbf{U}^{m-r}\mathbf{X}\mathbf{U}^{r-1} = \mathbf{C}$

We shall investigate the tensor equation

$$\sum_{r=1}^{m} \mathbf{U}^{m-r}\mathbf{X}\mathbf{U}^{r-1} = \mathbf{C}; \tag{14}$$

here $\mathbf{U}$ and $\mathbf{C}$ are given second-order tensors; $\mathbf{U}$ is symmetric and positive-definite; $m > 2$

is an integer. We set aside the case $m = 1$ because it is trivial. Since simple tensorial expressions for $\mathbf{T}^{(2)}$ and $\mathbf{T}^{(-2)}$ are already available in (6) and (7) respectively, the case $m = 2$ is irrelevant for our present purpose. Nevertheless we mention that our analysis below remains valid for $m = 1$ and $m = 2$. When $m = 2$, (14) assumes the form $\mathbf{UX}+\mathbf{XU} = \mathbf{C}$, which has been studied extensively in the literature. In particular the reader is invited to put $m = 2$ in our analysis below and compare it with the work of Sidoroff (1978) on the equation $\mathbf{UX}+\mathbf{XU} = \mathbf{C}$; both start by appealing to the same representation formula for the solution $\mathbf{X}$ [see (17) and (38) below], but they depart in taking different paths to arrive at the coefficients that appear in the representation formula.

The following assertions follow easily from eqn (14) and from the positive definiteness of $\mathbf{U}$:

(i) When $\mathbf{C} = \mathbf{O}$, the homogeneous equation has only a trivial solution.
(ii) Equation (14) has a unique solution $\mathbf{X}(\mathbf{U}, \mathbf{C})$, which is linear in $\mathbf{C}$ and is an isotropic function of $\mathbf{U}$ and $\mathbf{C}$.
(iii) The solution splits into two parts:

$$\mathbf{X}(\mathbf{U}, \mathbf{C}) = \mathbf{X}(\mathbf{U}, \tfrac{1}{2}(\mathbf{C}+\mathbf{C}^T)) + \mathbf{X}(\mathbf{U}, \tfrac{1}{2}(\mathbf{C}-\mathbf{C}^T)).$$

(iv) The right-hand side $\mathbf{C}$ and the solution $\mathbf{X}$ are simultaneously symmetric or skew-symmetric.

By decomposing the tensors $\mathbf{X}$ and $\mathbf{C}$ under the principal frame $\{\mathbf{N}_i\}$ of $\mathbf{U}$ as

$$\mathbf{X} = \sum_{i,j} x_{ij}\mathbf{N}_i \otimes \mathbf{N}_j, \quad \mathbf{C} = \sum_{i,j} c_{ij}\mathbf{N}_i \otimes \mathbf{N}_j, \tag{15}$$

we immediately obtain the unique solution of (14) in principal representation:

$$x_{ij} = \frac{c_{ij}}{\sum_{r=1}^{m} \lambda_i^{m-r}\lambda_j^{r-1}}. \tag{16}$$

Our objective in this paper is to find an expression of this solution in tensorial form.

Let us first consider the case of symmetric $\mathbf{C}$. By virtue of assertion (ii) and Rivlin's representation formula [cf. Rivlin (1955), eqn (10.17); Spencer (1971), Table V and Section 8; Wang (1970), p. 215], we may cast the (symmetric) solution (for symmetric $\mathbf{C}$) in the following form:

$$\mathbf{X}(\mathbf{U}, \mathbf{C}) = \alpha_1\mathbf{I}+\alpha_2\mathbf{U}+\alpha_3\mathbf{U}^2+\alpha_4\mathbf{C}+\alpha_5(\mathbf{UC}+\mathbf{CU})+\alpha_6(\mathbf{U}^2\mathbf{C}+\mathbf{CU}^2), \tag{17}$$

where α_4, α_5, α_6 are functions of the three principal invariants of $\mathbf{U}$:

$$\begin{aligned} \mathrm{I} &= \lambda_1+\lambda_2+\lambda_3, \\ \mathrm{II} &= \lambda_2\lambda_3+\lambda_3\lambda_1+\lambda_1\lambda_2, \\ \mathrm{III} &= \lambda_1\lambda_2\lambda_3, \end{aligned} \tag{18}$$

and $\alpha_1, \alpha_2, \alpha_3$ are functions of I, II, III and three common invariants of $\mathbf{U}$ and $\mathbf{C}$ that are linear in $\mathbf{C}$:

$$\begin{aligned} \mathrm{tr}\,\mathbf{C} &= c_{11}+c_{22}+c_{33}, \\ \mathrm{tr}\,(\mathbf{UC}) &= \lambda_1c_{11}+\lambda_2c_{22}+\lambda_3c_{33}, \\ \mathrm{tr}\,(\mathbf{U}^2\mathbf{C}) &= \lambda_1^2c_{11}+\lambda_2^2c_{22}+\lambda_3^2c_{33}. \end{aligned} \tag{19}$$

(19) can be written as

$$\begin{pmatrix} \operatorname{tr}\mathbf{C} \\ \operatorname{tr}(\mathbf{UC}) \\ \operatorname{tr}(\mathbf{U}^2\mathbf{C}) \end{pmatrix} = M^{\mathrm{T}} \begin{pmatrix} c_{11} \\ c_{22} \\ c_{33} \end{pmatrix}, \tag{20}$$

where

$$M := \begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 \\ 1 & \lambda_2 & \lambda_2^2 \\ 1 & \lambda_3 & \lambda_3^2 \end{pmatrix} \tag{21}$$

is the van der Monde matrix. Because of the linearity of $\mathbf{X}$ in $\mathbf{C}$, α_1, α_2 and α_3 must be linear combinations of $\operatorname{tr}\mathbf{C}$, $\operatorname{tr}(\mathbf{UC})$ and $\operatorname{tr}(\mathbf{U}^2\mathbf{C})$ and can be presented as

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = A \begin{pmatrix} \operatorname{tr}\mathbf{C} \\ \operatorname{tr}(\mathbf{UC}) \\ \operatorname{tr}(\mathbf{U}^2\mathbf{C}) \end{pmatrix} = AM^{\mathrm{T}} \begin{pmatrix} c_{11} \\ c_{22} \\ c_{33} \end{pmatrix}, \tag{22}$$

where the matrix

$$A := (\alpha_{ij}). \tag{23}$$

Comparing the componential form of (17) with (16), we have

$$\alpha_1 + \alpha_2\lambda_i + \alpha_3\lambda_i^2 + \alpha_4 c_{ii} + 2\alpha_5\lambda_i c_{ii} + 2\alpha_6\lambda_i^2 c_{ii} = \frac{c_{ii}}{m\lambda_i^{m-1}}, \quad i = 1, 2, 3, \tag{24}$$

$$\alpha_4 c_{ij} + \alpha_5(\lambda_i + \lambda_j)c_{ij} + \alpha_6(\lambda_i^2 + \lambda_j^2)c_{ij} = \frac{c_{ij}}{\sum_{r=1}^{m} \lambda_i^{m-r}\lambda_j^{r-1}}, \quad i \neq j. \tag{25}$$

We divide our further discussion into the case where all the eigenvalues of $\mathbf{U}$ are distinct and that which they are not. Henceforth we shall refer to the eigenvalues of $\mathbf{U}$ as the principal stretches.

Case 1. *All principal stretches are distinct*

Since $\mathbf{C}$ can be arbitrary and

$$\left(\sum_{r=1}^{m} \lambda_i^{m-r}\lambda_j^{r-1}\right)^{-1} = (\lambda_i - \lambda_j)/(\lambda_i^m - \lambda_j^m),$$

(25) is equivalent to

$$\begin{pmatrix} 1 & \lambda_2+\lambda_3 & \lambda_2^2+\lambda_3^2 \\ 1 & \lambda_3+\lambda_1 & \lambda_3^2+\lambda_1^2 \\ 1 & \lambda_1+\lambda_2 & \lambda_1^2+\lambda_2^2 \end{pmatrix} \begin{pmatrix} \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{pmatrix} = \begin{bmatrix} \dfrac{\lambda_2-\lambda_3}{\lambda_2^m-\lambda_3^m} \\ \dfrac{\lambda_3-\lambda_1}{\lambda_3^m-\lambda_1^m} \\ \dfrac{\lambda_1-\lambda_2}{\lambda_1^m-\lambda_2^m} \end{bmatrix}. \tag{26}$$

Since the determinant of the coefficient matrix

$$\Delta = (\lambda_2-\lambda_3)(\lambda_3-\lambda_1)(\lambda_1-\lambda_2) \neq 0, \tag{27}$$

(26) has a unique solution:

$$\begin{aligned}
\alpha_4 &= \frac{1}{\Delta}\sum_i \frac{(\mathrm{II}-\lambda_i^2)(\lambda_j-\lambda_k)^2}{\lambda_j^m-\lambda_k^m}, \\
\alpha_5 &= \frac{-1}{\Delta}\sum_i \frac{(\lambda_j+\lambda_k)(\lambda_j-\lambda_k)^2}{\lambda_j^m-\lambda_k^m}, \\
\alpha_6 &= \frac{1}{\Delta}\sum_i \frac{(\lambda_j-\lambda_k)^2}{\lambda_j^m-\lambda_k^m}.
\end{aligned} \tag{28}$$

Here the summation $\sum_i$ (or $\sum_{i'}$ later on) is carried out for all even permutations (i,j,k) [or (i',j',k')] of $(1,2,3)$. Substituting (28) back to (24), we have

$$\begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 \\ 1 & \lambda_2 & \lambda_2^2 \\ 1 & \lambda_3 & \lambda_3^2 \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} \mu_1 c_{11} \\ \mu_2 c_{22} \\ \mu_3 c_{33} \end{pmatrix}, \tag{29}$$

where

$$\mu_i = \frac{1}{m}\lambda_i^{1-m} - \alpha_4 - 2\alpha_5\lambda_i - 2\alpha_6\lambda_i^2. \tag{30}$$

Owing to (22), solving (29) is equivalent to finding the matrix A defined in (23). In fact, by denoting the matrices

$$E = \mathrm{diag}(1,1,1), \quad U = \mathrm{diag}(\lambda_1,\lambda_2,\lambda_3), \tag{31}$$

(29) can be written in matrix form as follows:

$$M\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \left(\frac{1}{m}U^{1-m} - \alpha_4 E - 2\alpha_5 U - 2\alpha_6 U^2\right)\begin{pmatrix} c_{11} \\ c_{22} \\ c_{33} \end{pmatrix}. \tag{32}$$

Substituting (22) into the left-hand side of (32), in virtue of the arbitrariness of $\{c_{ii}\}$, we have

$$MAM^{T} = \frac{1}{m}U^{1-m} - \alpha_4 E - 2\alpha_5 U - 2\alpha_6 U^2$$

and, consequently,

$$A = \frac{1}{m}A^{(1-m)} - \alpha_4 A^{(0)} - 2\alpha_5 A^{(1)} - 2\alpha_6 A^{(2)}, \tag{33}$$

where

$$A^{(\xi)} := M^{-1}U^{\xi}M^{-T}, \quad \xi = 1-m, 0, 1, 2. \tag{34}$$

It can be seen that A is symmetric. Having

$$M^{-1} = \frac{1}{\Delta}\begin{pmatrix} \lambda_2\lambda_3(\lambda_3-\lambda_2) & \lambda_3\lambda_1(\lambda_1-\lambda_3) & \lambda_1\lambda_2(\lambda_2-\lambda_1) \\ \lambda_2^2-\lambda_3^2 & \lambda_3^2-\lambda_1^2 & \lambda_1^2-\lambda_2^2 \\ \lambda_3-\lambda_2 & \lambda_1-\lambda_3 & \lambda_2-\lambda_1 \end{pmatrix} \tag{35}$$

at hand, we can easily use (34) to calculate $A^{(5)}$ and to get the matrix A with the following entries:

$$\alpha_{11} = \frac{1}{\Delta^2}\sum_i B_i\lambda_j^2\lambda_k^2(\lambda_j-\lambda_k)^2,$$

$$\alpha_{12} = \frac{-1}{\Delta^2}\sum_i B_i\lambda_j\lambda_k(\lambda_j-\lambda_k)(\lambda_j^2-\lambda_k^2),$$

$$\alpha_{13} = \frac{1}{\Delta^2}\sum_i B_i\lambda_j\lambda_k(\lambda_j-\lambda_k)^2,$$

$$\alpha_{22} = \frac{1}{\Delta^2}\sum_i B_i(\lambda_j^2-\lambda_k^2)^2,$$

$$\alpha_{23} = \frac{-1}{\Delta^2}\sum_i B_i(\lambda_j-\lambda_k)(\lambda_j^2-\lambda_k^2),$$

$$\alpha_{33} = \frac{1}{\Delta^2}\sum_i B_i(\lambda_j-\lambda_k)^2, \tag{36}$$

where

$$B_i = \frac{1}{m\lambda_i^{m-1}} - \frac{1}{\Delta}\sum_{i'}[(\mathrm{II}-\lambda_{i'}^2) - 2(\lambda_{j'}+\lambda_{k'})\lambda_i + 2\lambda_i^2]\,\frac{(\lambda_{j'}-\lambda_{k'})^2}{\lambda_{j'}^m-\lambda_{k'}^m}. \tag{37}$$

Thus, an intrinsic expression for the solution of (14) is

$$\begin{aligned} \mathbf{X}(\mathbf{U},\mathbf{C}) = {} & [\alpha_{11}\,\mathrm{tr}\,\mathbf{C}+\alpha_{12}\,\mathrm{tr}\,(\mathbf{UC})+\alpha_{13}\,\mathrm{tr}\,(\mathbf{U}^2\mathbf{C})]\mathbf{I} \\ & +[\alpha_{12}\,\mathrm{tr}\,\mathbf{C}+\alpha_{22}\,\mathrm{tr}\,(\mathbf{UC})+\alpha_{23}\,\mathrm{tr}\,(\mathbf{U}^2\mathbf{C})]\mathbf{U} \\ & +[\alpha_{13}\,\mathrm{tr}\,\mathbf{C}+\alpha_{23}\,\mathrm{tr}\,(\mathbf{UC})+\alpha_{33}\,\mathrm{tr}\,(\mathbf{U}^2\mathbf{C})]\mathbf{U}^2 \\ & +\alpha_4\mathbf{C}+\alpha_5(\mathbf{UC}+\mathbf{CU})+\alpha_6(\mathbf{U}^2\mathbf{C}+\mathbf{CU}^2). \end{aligned} \tag{38}$$

Case 2. Two or all of the principal stretches are coalescent

A glance at (24) and (25) reveals that the solution set of the six simultaneous linear equations for α_i ($i = 1, 2, \ldots, 6$) is infinite. Each solution 6-tuple of α_i, when substituted into (17), will give a representation of the solution $\mathbf{X}(\mathbf{U},\mathbf{C})$ of (14). Nevertheless, as we shall see, the right-hand sides of (28) and (36) may be extended by continuity so that their domains of definition will include the possibility of coalescent stretches. Moreover, when we use the values of α_k ($k = 4, 5, 6$) and α_{ij} ($i, j = 1, 2, 3$) thus determined from (28) and (36), eqn (38) still delivers *one* representation of the unique solution $\mathbf{X}(\mathbf{U},\mathbf{C})$ in tensorial form.

First let us revert to the case of distinct principal stretches and examine (25) more closely. By Cramer's rule,

$$
\alpha_4 = \frac{1}{\Delta}\begin{vmatrix} \dfrac{1}{\sum_{r=1}^{m} \lambda_2^{m-r}\lambda_3^{r-1}} & \lambda_2+\lambda_3 & \lambda_2^2+\lambda_3^2 \\ \dfrac{1}{\sum_{r=1}^{m} \lambda_3^{m-r}\lambda_1^{r-1}} & \lambda_3+\lambda_1 & \lambda_3^2+\lambda_1^2 \\ \dfrac{1}{\sum_{r=1}^{m} \lambda_1^{m-r}\lambda_2^{r-1}} & \lambda_1+\lambda_2 & \lambda_1^2+\lambda_2^2 \end{vmatrix} . \tag{39}
$$

It is clear that when we expand the numerator determinant of α_4 we obtain a rational function of λ_1, λ_2, λ_3. For definiteness, let this rational function be denoted by $f(\lambda_1, \lambda_2, \lambda_3)/\Box$, where f is a polynomial in λ_1, λ_2, λ_3, and

$$
\Box(\lambda_1, \lambda_2, \lambda_3) := \prod_i \left(\sum_{r=1}^{m} \lambda_j^{m-r}\lambda_k^{r-1} \right) > 0 ; \tag{40}
$$

here the product is meant to be carried out for all even permutations (i, j, k) of $(1, 2, 3)$. If we set $\lambda_1 = \lambda_2$ in the numerator determinant of (39), the first two rows of the determinant will be the same; hence we conclude that $f(\lambda_2, \lambda_2, \lambda_3)/\Box(\lambda_2, \lambda_2, \lambda_3) = 0$. Since $\Box(\lambda_2, \lambda_2, \lambda_3) > 0$, we must have $f(\lambda_2, \lambda_2, \lambda_3) = 0$. If we regard λ_2 and λ_3 as given, f is simply a polynomial in λ_1. By the remainder theorem, $f(\lambda_2, \lambda_2, \lambda_3) = 0$ implies that $\lambda_1 - \lambda_2$ is a factor of f. Similarly we can prove that $\lambda_2 - \lambda_3$ and $\lambda_3 - \lambda_1$ are also factors of f. Hence $f = \Delta \cdot g$, where g is a polynomial in λ_1, λ_2, λ_3. It follows that $\alpha_4 = g/\Box$, which is clearly well defined even for the present case of coalescent stretches. The discussion for α_5 and α_6 is similar.

After the values of α_4, α_5 and α_6 are determined as above, (24) becomes a system of simultaneous linear equations for the unknowns α_1, α_2 and α_3. Again, we start by considering the case of distinct principal stretches. Then, by Cramer's rule,

$$
\alpha_1 = \frac{1}{\Delta}\begin{vmatrix} c_{11}\left(\dfrac{1}{m\lambda_1^{m-1}} - \alpha_4 - 2\alpha_5\lambda_1 - 2\alpha_6\lambda_1^2\right) & \lambda_1 & \lambda_1^2 \\ c_{22}\left(\dfrac{1}{m\lambda_2^{m-1}} - \alpha_4 - 2\alpha_5\lambda_2 - 2\alpha_6\lambda_2^2\right) & \lambda_2 & \lambda_2^2 \\ c_{33}\left(\dfrac{1}{m\lambda_3^{m-1}} - \alpha_4 - 2\alpha_5\lambda_3 - 2\alpha_6\lambda_3^2\right) & \lambda_3 & \lambda_3^2 \end{vmatrix} . \tag{41}
$$

Since α_4, α_5 and α_6 satisfy (25), if we set $\lambda_1 = \lambda_2$, we obtain the equation

$$
\alpha_4 + 2\lambda_2\alpha_5 + 2\lambda_2^2\alpha_6 = \frac{1}{m\lambda_2^{m-1}} . \tag{42}
$$

It follows that the numerator determinant of α_1 vanishes when $\lambda_1 = \lambda_2$. This determinant is on expansion a rational function $P(\lambda_1, \lambda_2, \lambda_3)/Q(\lambda_1, \lambda_2, \lambda_3)$, where the denominator polynomial

$$
Q = m^3\lambda_1^{m-1}\lambda_2^{m-1}\lambda_3^{m-1}\Box \tag{43}
$$

is always positive. Since $P(\lambda_2, \lambda_2, \lambda_3)/Q(\lambda_2, \lambda_2, \lambda_3) = 0$ and $Q > 0$, we conclude that $\lambda_1 - \lambda_2$ is a factor of the polynomial P. Similarly we can prove that $\lambda_2 - \lambda_3$ and $\lambda_3 - \lambda_1$ are also factors of P. In other words $P = \Delta \cdot h_1$, where $h_1(\lambda_1, \lambda_2, \lambda_3)$ is a polynomial in λ_1, λ_2 and λ_3, and we conclude that $\alpha_1 = h_1/Q$. Similarly, $\alpha_2 = h_2/Q$ and $\alpha_3 = h_3/Q$, where h_2 and h_3 are polynomials in λ_1, λ_2 and λ_3. In fact, a closer examination of (41) and of its counterparts for α_2, α_3 reveals that

$$\alpha_i = \frac{1}{Q}(h_{i1}c_{11} + h_{i2}c_{22} + h_{i3}c_{33}), \quad i = 1, 2, 3, \tag{44}$$

where h_{ij} are polynomials in λ_1, λ_2 and λ_3.

Let $\beta_{ij} := h_{ij}/Q$. We see from (22) and (44) that the matrix

$$B := (\beta_{ij}) = AM^{\mathrm{T}}. \tag{45}$$

Since both h_{ij} and Q are defined even for coalescent stretches, so do the entries β_{ij} of the matrix B. Indeed we claim that

$$\beta_{ij} = \beta_{ik}, \quad \text{if} \quad \lambda_j = \lambda_k. \tag{46}$$

Let us prove the preceding assertion for β_{11} and β_{12}. Proofs for other cases are similar. When the principal stretches are all distinct, β_{11} and β_{12} are given by the formulae:

$$\beta_{11}(\lambda_1, \lambda_2, \lambda_3) = \frac{1}{\Delta}\lambda_3\lambda_2(\lambda_3 - \lambda_2)\left(\frac{1}{m\lambda_1^{m-1}} - \alpha_4 - 2\alpha_5\lambda_1 - 2\alpha_6\lambda_1^2\right),$$

$$\beta_{12}(\lambda_1, \lambda_2, \lambda_3) = \frac{1}{\Delta}\lambda_1\lambda_3(\lambda_1 - \lambda_3)\left(\frac{1}{m\lambda_2^{m-1}} - \alpha_4 - 2\alpha_5\lambda_2 - 2\alpha_6\lambda_2^2\right). \tag{47}$$

The system of simultaneous equations (25) remains invariant under the exchange of λ_i and λ_j $(i \neq j)$. Since we are discussing the case of distinct principal stretches for the moment, we conclude from the uniqueness of the solution of (25) that α_4, α_5 and α_6 are symmetric functions of λ_1, λ_2 and λ_3. It follows that

$$\beta_{11}(\lambda_2, \lambda_1, \lambda_3) = \beta_{12}(\lambda_1, \lambda_2, \lambda_3) \tag{48}$$

when the principal stretches are distinct.

As mentioned earlier, we may extend the domain of definition of β_{11} and β_{12} to include the possibility of coalescent stretches. In fact, $\beta_{11} = h_{11}/Q$ and $\beta_{12} = h_{12}/Q$, where h_{11} and h_{12} are polynomials in λ_1, λ_2 and λ_3, and $Q > 0$ is defined in (43). Since Q is a symmetric function of the principal stretches, we conclude from (48) that $h_{11}(\lambda_2, \lambda_1, \lambda_3) = h_{12}(\lambda_1, \lambda_2, \lambda_3)$, which implies $h_{11}(\lambda_2, \lambda_2, \lambda_3) = h_{12}(\lambda_2, \lambda_2, \lambda_3)$. It follows that $\beta_{11} = \beta_{12}$ if $\lambda_1 = \lambda_2$.

We obtain the matrix $A := (\alpha_{ij})$ by multiplying the matrix B by $M^{-\mathrm{T}}$. We may write down the entries of A by using (35) and (45). For instance,

$$\alpha_{11} = \frac{1}{\Delta}(\lambda_2\lambda_3(\lambda_3 - \lambda_2)\beta_{11} + \lambda_3\lambda_1(\lambda_1 - \lambda_3)\beta_{12} + \lambda_1\lambda_2(\lambda_2 - \lambda_1)\beta_{13}). \tag{49}$$

The numerator of α_{11} in (49) is a rational function of λ_1, λ_2 and λ_3 whose denominator polynomial is always positive. If we set $\lambda_1 = \lambda_2$, we see from (46) that this numerator vanishes. Hence we conclude that $\lambda_1 - \lambda_2$ is a factor of the numerator polynomial of the rational function in question. Similarly $\lambda_2 - \lambda_3$ and $\lambda_3 - \lambda_1$ are also factors. On cancellation of Δ from its numerator and denominator, α_{11} is expressed as a rational function whose denominator polynomial is always positive. Therefore we may extend α_{11} by continuity so that its domain of definition includes the present case of coalescent stretches. By similar arguments, we can reach the same conclusion for the other α_{ij}.

For definiteness let $\bar{\alpha}_{ij}(\lambda_1, \lambda_2, \lambda_3)$ and $\bar{\alpha}_k(\lambda_1, \lambda_2, \lambda_3)$ $(i, j = 1, 2, 3;\ k = 4, 5, 6)$ be the functions that result from (36) and (28), respectively, after we extend the functions α_{ij} and α_k by continuity to include the possibility of coalescent stretches. We claim that the coefficients $\bar{\alpha}_{ij}$ and $\bar{\alpha}_k$ always deliver *one* representation of $\mathbf{X}(\mathbf{U}, \mathbf{C})$ via (38) even if two or all of the principal stretches coalesce.

For simplicity let us rewrite (24) and (25) as

$$F_l(\alpha_{11}, \ldots, \alpha_{33}, \alpha_4, \alpha_5, \alpha_6 ; \lambda_1, \lambda_2, \lambda_3) = 0, \quad l = 1, 2, 3,$$
$$G_m(\alpha_4, \alpha_5, \alpha_6 ; \lambda_1, \lambda_2, \lambda_3) = 0, \quad m = 4, 5, 6, \tag{50}$$

respectively; here, in rewriting (24), we have used (22) to replace α_1, α_2 and α_3 by α_{ij} $(i, j = 1, 2, 3)$. The functions F_l and G_m are continuous in all their arguments.

Let us now consider the case where exactly two principal stretches are equal. Without loss of generality, suppose $\lambda_1 = \lambda_2 \neq \lambda_3$. Let $\{\lambda_1^{(n)}\}$ be a sequence such that $\lim_{n \to \infty} \lambda_1^{(n)} = \lambda_2$, and $\lambda_1^{(n)} \neq \lambda_2$, $\lambda_1^{(n)} \neq \lambda_3$ for each n. Since $\lambda_1^{(n)}$, λ_2 and λ_3 are distinct, we know from our previous discussion that $\bar{\alpha}_{ij}(\lambda_1^{(n)}, \lambda_2, \lambda_3)$ and $\bar{\alpha}_k(\lambda_1^{(n)}, \lambda_2, \lambda_3)$ satisfy the equations

$$F_l(\bar{\alpha}_{11}, \ldots, \bar{\alpha}_{33}, \bar{\alpha}_4, \bar{\alpha}_5, \bar{\alpha}_6 ; \lambda_1^{(n)}, \lambda_2, \lambda_3) = 0, \quad G_m(\bar{\alpha}_4, \bar{\alpha}_5, \bar{\alpha}_6 ; \lambda_1^{(n)}, \lambda_2, \lambda_3) = 0, \tag{51}$$

for each n. Since F_l, G_m, $\bar{\alpha}_{ij}$ and $\bar{\alpha}_k$ are all continuous in their respective arguments, we conclude that as $n \to \infty$, $\bar{\alpha}_{ij}(\lambda_2, \lambda_2, \lambda_3)$ and $\bar{\alpha}_k(\lambda_2, \lambda_2, \lambda_3)$ satisfy the equations

$$F_l(\bar{\alpha}_{11}, \ldots, \bar{\alpha}_{33}, \bar{\alpha}_4, \bar{\alpha}_5, \bar{\alpha}_6 ; \lambda_2, \lambda_2, \lambda_3) = 0, \quad G_m(\bar{\alpha}_4, \bar{\alpha}_5, \bar{\alpha}_6 ; \lambda_2, \lambda_2, \lambda_3) = 0. \tag{52}$$

Hence $\bar{\alpha}_{ij}$ and $\bar{\alpha}_k$ still deliver via (38) one representation for the unique solution $\mathbf{X}(\mathbf{U}, \mathbf{C})$ of (14).

It is clear that when the principal stretches all coalesce we can prove our assertion in a similar way.

We summarize our finding in the following:

Theorem. Let $\mathbf{U}$ *and* $\mathbf{C}$ *be given symmetric second-order tensors, where* $\mathbf{U}$ *is positive definite. Let* $m > 2$ *be an integer. When the principal stretches (i.e., the eigenvalues of* $\mathbf{U}$*) are distinct, eqn* (38), *in which the coefficients* α_k $(k = 4, 5, 6)$ *and* α_{ij} $(i, j = 1, 2, 3)$ *are given by* (28) *and* (36), *respectively, provides an explicit tensorial representation for the unique solution* $\mathbf{X}(\mathbf{U}, \mathbf{C})$ *of eqn* (14). *The coefficients* α_k *and* α_{ij}, *which are functions of the principal stretches* λ_1, λ_2 *and* λ_3, *can be extended by continuity so that their common domain of definition includes the possibility of coalescent stretches. When the coefficients are thus extended, eqn* (38) *always delivers one representation of* $\mathbf{X}(\mathbf{U}, \mathbf{C})$ *in tensorial form, irrespective of whether the principal stretches are distinct or not.*

For a skew-symmetric $\mathbf{C}$, the skew-symmetric solution of (14) can be cast in the form [cf. Spencer (1971), Table VIII and Section 8]:

$$\mathbf{X}(\mathbf{U}, \mathbf{C}) = \beta_4 \mathbf{C} + \beta_5(\mathbf{UC} + \mathbf{CU}) + \beta_6(\mathbf{U}^2\mathbf{C} + \mathbf{CU}^2), \tag{53}$$

where β_4, β_5 and β_6 are functions of I, II, III. The form of (53) and the last three terms of (17) are similar, and the system of equations for determining β_4, β_5, β_6 is the same as (25) for α_4, α_5, α_6. Therefore, when the principal stretches are distinct, β_4, β_5, β_6 are given also by (28); moreover, these coefficients can be extended by continuity to allow for the possibility of coalescent stretches. Hence we arrive at the following:

Corollary. For a skew-symmetric $\mathbf{C}$, *all the conclusions of the preceding theorem stand, except that the coefficients* α_{ij} *should be taken as null.*

By assertion (iii) at the beginning of this section, the theorem and corollary above will together provide a tensorial expression for the unique solution $\mathbf{X}(\mathbf{U}, \mathbf{C})$ of eqn (14) for any given second-order tensor C. Using an algorithm (cf. Appendix) based on the fundamental theorem of symmetric polynomials, we can express all the coefficients α_{11}, $\alpha_{12}, \ldots, \alpha_{33}$, α_4, α_5, α_6 of the solution (38) in terms of the principal invariants I, II, III of $\mathbf{U}$. We shall illustrate the entire procedure by finding an explicit intrinsic expression for the conjugate stress $\mathbf{T}^{(-3)}$.

3. EXPRESSION FOR CONJUGATE STRESS $\mathbf{T}^{(-3)}$

By (12) the stress $\mathbf{T}^{(-3)}$ conjugate to the strain $\mathbf{E}^{(-3)} = \frac{1}{3}(\mathbf{I}-\mathbf{U}^{-3})$ is the unique solution of the tensor equation

$$\mathbf{U}^2\mathbf{X}+\mathbf{U}\mathbf{X}\mathbf{U}+\mathbf{X}\mathbf{U}^2 = 3\mathbf{U}^2\mathbf{T}^{(-1)}\mathbf{U}^2. \tag{54}$$

Denoting $\mathbf{T}^{(-1)} = \sum_{i,j} t_{ij}\mathbf{N}_i \otimes \mathbf{N}_j$, we obtain the solution of (54) in principal representation:

$$x_{ij} = \frac{3\lambda_i^2\lambda_j^2 t_{ij}}{\lambda_i^2+\lambda_i\lambda_j+\lambda_j^2}. \tag{55}$$

An intrinsic expression of $\mathbf{X}$ is

$$\begin{aligned}
\mathbf{X}(\mathbf{U},\mathbf{T}^{(-1)}) &= \gamma_1\mathbf{I}+\gamma_2\mathbf{U}+\gamma_3\mathbf{U}^2+\gamma_4\mathbf{T}^{(-1)}+\gamma_5(\mathbf{U}\mathbf{T}^{(-1)}+\mathbf{T}^{(-1)}\mathbf{U}) \\
&\quad +\gamma_6(\mathbf{U}^2\mathbf{T}^{(-1)}+\mathbf{T}^{(-1)}\mathbf{U}^2) \\
&= [\gamma_{11}\,\mathrm{tr}\,\mathbf{T}^{(-1)}+\gamma_{12}\,\mathrm{tr}\,(\mathbf{U}\mathbf{T}^{(-1)})+\gamma_{13}\,\mathrm{tr}\,(\mathbf{U}^2\mathbf{T}^{(-1)})]\mathbf{I} \\
&\quad +[\gamma_{12}\,\mathrm{tr}\,\mathbf{T}^{(-1)}+\gamma_{22}\,\mathrm{tr}\,(\mathbf{U}\mathbf{T}^{(-1)})+\gamma_{23}\,\mathrm{tr}\,(\mathbf{U}^2\mathbf{T}^{(-1)})]\mathbf{U} \\
&\quad +[\gamma_{13}\,\mathrm{tr}\,\mathbf{T}^{(-1)}+\gamma_{23}\,\mathrm{tr}\,(\mathbf{U}\mathbf{T}^{(-1)})+\gamma_{33}\,\mathrm{tr}\,(\mathbf{U}^2\mathbf{T}^{(-1)})]\mathbf{U}^2 \\
&\quad +\gamma_4\mathbf{T}^{(-1)}+\gamma_5(\mathbf{U}\mathbf{T}^{(-1)}+\mathbf{T}^{(-1)}\mathbf{U})+\gamma_6(\mathbf{U}^2\mathbf{T}^{(-1)}+\mathbf{T}^{(-1)}\mathbf{U}^2).
\end{aligned} \tag{56}$$

The equations for determining the coefficients $\gamma_1, \gamma_2, \ldots, \gamma_6$ are

$$\gamma_1+\gamma_2\lambda_i+\gamma_3\lambda_i^2+\gamma_4 t_{ii}+2\gamma_5\lambda_i t_{ii}+2\gamma_6\lambda_i^2 t_{ii} = \lambda_i^2 t_{ii}, \quad i = 1,2,3, \tag{57}$$

$$\gamma_4 t_{ij}+\gamma_5(\lambda_i+\lambda_j)t_{ij}+\gamma_6(\lambda_i^2+\lambda_j^2)t_{ij} = \frac{3\lambda_i^2\lambda_j^2 t_{ij}}{\lambda_i^2+\lambda_i\lambda_j+\lambda_j^2}, \quad i \neq j. \tag{58}$$

To start with, we proceed as if the principal stretches are distinct. Solving the system equivalent to (58), namely

$$\begin{pmatrix} 1 & \lambda_2+\lambda_3 & \lambda_2^2+\lambda_3^2 \\ 1 & \lambda_3+\lambda_1 & \lambda_3^2+\lambda_1^2 \\ 1 & \lambda_1+\lambda_2 & \lambda_1^2+\lambda_2^2 \end{pmatrix}\begin{pmatrix} \gamma_4 \\ \gamma_5 \\ \gamma_6 \end{pmatrix} = 3\begin{bmatrix} \dfrac{\lambda_2^2\lambda_3^2}{\lambda_2^2+\lambda_2\lambda_3+\lambda_3^2} \\ \dfrac{\lambda_3^2\lambda_1^2}{\lambda_3^2+\lambda_3\lambda_1+\lambda_1^2} \\ \dfrac{\lambda_1^2\lambda_2^2}{\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2} \end{bmatrix},$$

we obtain the solution γ_4, γ_5, γ_6 in the form of rational functions of λ_1, λ_2, λ_3. Cancelling the common skew-symmetric factor Δ in the numerators and denominators, we use the algorithm in the Appendix to get

$$\gamma_4 = \frac{3}{\Box}(\mathrm{I\,II^2\,III + I^2\,III^2 - II^4}),$$

$$\gamma_5 = \frac{3}{\Box}(\mathrm{I\,II^3 - I^2\,II\,III - II^2\,III}),$$

$$\gamma_6 = \frac{3}{\Box}(\mathrm{I\,II\,III - II^3}), \tag{59}$$

where

$$\Box := \prod_i (\lambda_j^2 + \lambda_j\lambda_k + \lambda_k^2) = \mathrm{I^2\,II^2 - I^3\,III - II^3} > 0. \tag{60}$$

Substituting (59) into (57) and following the procedure described in Section 2, we obtain the expression for the matrix:

$$\Gamma := (\gamma_{ij}) = -\gamma_4 A^{(0)} - 2\gamma_5 A^{(1)} + (1 - 2\gamma_6)A^{(2)}, \tag{61}$$

where $A^{(\xi)} = M^{-1}U^{\xi}M^{-T}$ for $\xi = 0, 1, 2$ [cf. (34)]. Let $\mathring{\Gamma} := \Delta^2\Box\Gamma = (\mathring{\gamma}_{ij})$. The entries of $\mathring{\Gamma}$ are

$$\begin{aligned}
\mathring{\gamma}_{11} = {} & 81\,\mathrm{I\,II^2\,III^3} + 66\,\mathrm{I\,II^5\,III} - 50\,\mathrm{I^2\,II^3\,III^2} + 3\,\mathrm{I^2\,II^6} + 27\,\mathrm{I^2\,III^4} - 18\,\mathrm{I^3\,II\,III^3} \\
& - 15\,\mathrm{I^3\,II^4\,III} + 11\,\mathrm{I^4\,II^2\,III^2} + 4\,\mathrm{I^5\,III^3} - 81\,\mathrm{II^4\,III^2} - 12\,\mathrm{II^7}, \\
\mathring{\gamma}_{12} = {} & 117\,\mathrm{I\,II^3\,III^2} + 12\,\mathrm{I\,II^6} - 54\,\mathrm{I^2\,II\,III^3} - 60\,\mathrm{I^2\,II^4\,III} + 28\,\mathrm{I^3\,II^2\,III^2} - 3\,\mathrm{I^3\,II^5} \\
& + 14\,\mathrm{I^4\,II^3\,III} - 8\,\mathrm{I^5\,II\,III^2} - 54\,\mathrm{II^2\,III^3} - 8\,\mathrm{II^5\,III}, \\
\mathring{\gamma}_{13} = {} & 54\,\mathrm{I\,II\,III^3} + 62\,\mathrm{I\,II^4\,III} - 36\,\mathrm{I^2\,II^2\,III^2} + 3\,\mathrm{I^2\,II^5} - 14\,\mathrm{I^3\,II^3\,III} + 8\,\mathrm{I^4\,II\,III^2} \\
& - 81\,\mathrm{II^3\,III^2} - 12\,\mathrm{II^6}, \\
\mathring{\gamma}_{22} = {} & 54\,\mathrm{I\,II\,III^3} + 26\,\mathrm{I\,II^4\,III} - 63\,\mathrm{I^2\,II^2\,III^2} - 3\,\mathrm{I^2\,II^5} + 8\,\mathrm{I^3\,II^3\,III} - 27\,\mathrm{I^3\,III^3} + 26\,\mathrm{I^4\,II\,III^2} \\
& + \mathrm{I^4\,II^4} - 3\,\mathrm{I^5\,II^2\,III} - 4\,\mathrm{I^6\,III^2} - 27\,\mathrm{II^3\,III^2} - 4\,\mathrm{II^6}, \\
\mathring{\gamma}_{23} = {} & 27\,\mathrm{I\,II^2\,III^2} + 4\,\mathrm{I\,II^5} - 14\,\mathrm{I^2\,II^3\,III} + 27\,\mathrm{I^2\,III^3} - 18\,\mathrm{I^3\,II\,III^2} - \mathrm{I^3\,II^4} \\
& + 3\,\mathrm{I^4\,II^2\,III} + 4\,\mathrm{I^5\,III^2}, \\
\mathring{\gamma}_{33} = {} & 36\,\mathrm{I\,II^3\,III} + 2\,\mathrm{I^2\,II^4} - 8\,\mathrm{I^3\,II^2\,III} - 54\,\mathrm{II^2\,III^2} - 8\,\mathrm{II^5}.
\end{aligned} \tag{62}$$

Extracting the factor

$$\begin{aligned}
\Delta^2 &= (\lambda_2 - \lambda_3)^2(\lambda_3 - \lambda_1)^2(\lambda_1 - \lambda_2)^2 \\
&= 18\,\mathrm{I\,II\,III} + \mathrm{I^2\,II^2} - 4\,\mathrm{I^3\,III} - 4\,\mathrm{II^3} - 27\,\mathrm{III^2}
\end{aligned} \tag{63}$$

from the above expressions for $\mathring{\gamma}_{ij}$, we arrive at the intrinsic expression

$$\begin{aligned}
\mathbf{T}^{(-3)} = \frac{1}{\Box}\{ & [(-3\,\mathrm{I\,II^2\,III} - \mathrm{I^2\,III^2} + 3\,\mathrm{II^4})\,\mathrm{tr}\,\mathbf{T}^{(-1)} + (-3\,\mathrm{I\,II^3} + 2\mathrm{I^2\,II\,III} + 2\,\mathrm{II^2\,III})\,\mathrm{tr}\,(\mathbf{UT}^{(-1)}) \\
& + (-2\,\mathrm{I\,II\,III} + 3\,\mathrm{II^3})\,\mathrm{tr}\,(\mathbf{U}^2\mathbf{T}^{(-1)})]\mathbf{I} + [(-3\,\mathrm{I\,II^3} + 2\,\mathrm{I^2\,II\,III} + 2\,\mathrm{II^2\,III})\,\mathrm{tr}\,\mathbf{T}^{(-1)} \\
& + (-2\,\mathrm{I\,II\,III} + \mathrm{I^2\,II^2} + \mathrm{I^3\,III} + \mathrm{II^3})\,\mathrm{tr}\,(\mathbf{UT}^{(-1)}) - (\mathrm{I\,II^2} + \mathrm{I^2\,III})\,\mathrm{tr}\,(\mathbf{U}^2\mathbf{T}^{(-1)})]\mathbf{U} \\
& + [(-2\,\mathrm{I\,II\,III} + 3\,\mathrm{II^3})\,\mathrm{tr}\,\mathbf{T}^{(-1)} - (\mathrm{I\,II^2} + \mathrm{I^2\,III})\,\mathrm{tr}\,(\mathbf{UT}^{(-1)}) + 2\,\mathrm{II^2}\,\mathrm{tr}\,(\mathbf{U}^2\mathbf{T}^{(-1)})]\mathbf{U}^2 \\
& + 3[(\mathrm{I\,II^2\,III} + \mathrm{I^2\,III^2} - \mathrm{II^4})\mathbf{T}^{(-1)} + (\mathrm{I\,II^3} - \mathrm{I^2\,II\,III} - \mathrm{II^2\,III})(\mathbf{UT}^{(-1)} + \mathbf{T}^{(-1)}\mathbf{U}) \\
& + (\mathrm{I\,II\,III} - \mathrm{II^3})(\mathbf{U}^2\mathbf{T}^{(-1)} + \mathbf{T}^{(-1)}\mathbf{U}^2)]\}.
\end{aligned} \tag{64}$$

which is valid for any given $\mathbf{U}$ and $\mathbf{T}^{(-1)}$.

4. CONCLUDING REMARKS

For each integer m with $|m| > 2$, the method proposed above delivers an intrinsic expression for the stress $\mathbf{T}^{(m)}$ conjugate to the strain measure $\mathbf{E}^{(m)}$ in the Seth–Hill class. The invested labor, however, will increase rapidly with increasing $|m|$. Symbolic computation could be useful here, because all operations in this method are algebraic.

Acknowledgements—The collaborative research reported here was carried out when Guo held a visiting professorship at the Department of Mathematics, University of Kentucky, Lexington. That professorship was supported in part with funds from the National Science Foundation (Grant No. RII-8610671) and the Commonwealth of Kentucky through the Kentucky EPSCoR Program. This research was supported in part by the University of Kentucky under Faculty Grant No. 2-04449.

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APPENDIX

Throughout this paper we make use of an algorithm that recasts a symmetric polynomial of the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of the symmetric tensor $\mathbf{U}$ in terms of its principal invariants I, II, III. As an illustration, we demonstrate this algorithm for

$$\Delta^2 = (\lambda_2-\lambda_3)^2(\lambda_3-\lambda_1)^2(\lambda_1-\lambda_2)^2,$$

which is a homogeneous symmetric polynomial of degree six. In this instance the algorithm in question proceeds as follows:

(i) Write down all the elementary homogeneous polynomials of degree six in λ_1, λ_2 and λ_3, namely,

$$A = \lambda_1^2\lambda_2^2\lambda_3^2,$$

$$B = \sum_i (\lambda_i\lambda_j^2\lambda_k^3 + \lambda_i\lambda_j^3\lambda_k^2),$$

$$C = \sum_i \lambda_i\lambda_j\lambda_k^4,$$

$$D = \sum_i \lambda_j^3\lambda_k^3,$$

$$E = \sum_i (\lambda_j^2\lambda_k^4 + \lambda_j^4\lambda_k^2),$$

$$F = \sum_i (\lambda_j\lambda_k^5 + \lambda_j^5\lambda_k),$$

$$G = \sum_i \lambda_i^6.$$

(ii) Express in terms of the elementary polynomials each monomial of I, II, III that is of degree six in $\lambda_1, \lambda_2, \lambda_3$:

$$
\begin{aligned}
\mathrm{III}^2 &= A,\\
\mathrm{I\,II\,III} &= 3A+B,\\
\mathrm{I}^3\,\mathrm{III} &= 6A+3B+C,\\
\mathrm{II}^3 &= 6A+3B+D,\\
\mathrm{I}^2\,\mathrm{II}^2 &= 15A+8B+2C+2D+E,\\
\mathrm{I}^4\,\mathrm{II} &= 36A+22B+9C+6D+4E+F,\\
\mathrm{I}^6 &= 90A+60B+30C+20D+15E+6F+G.
\end{aligned}
$$

(iii) Solve the above linear system with a triangular coefficient matrix to write the elementary polynomials in terms of I, II and III:

$$
\begin{aligned}
A &= \mathrm{III}^2,\\
B &= \mathrm{I\,II\,III} - 3\,\mathrm{III}^2,\\
C &= \mathrm{I}^3\,\mathrm{III} - 3\,\mathrm{I\,II\,III} + 3\,\mathrm{III}^2,\\
D &= -3\,\mathrm{I\,II\,III} + \mathrm{II}^3 + 3\,\mathrm{III}^2,\\
E &= -2\,\mathrm{I}^3\,\mathrm{III} + \mathrm{I}^2\,\mathrm{II}^2 + 4\,\mathrm{I\,II\,III} - 2\,\mathrm{II}^3 - 3\,\mathrm{III}^2,\\
F &= \mathrm{I}^4\,\mathrm{II} - \mathrm{I}^3\,\mathrm{III} - 4\,\mathrm{I}^2\,\mathrm{II}^2 + 7\,\mathrm{I\,II\,III} + 2\,\mathrm{II}^3 - 3\,\mathrm{III}^2,\\
G &= \mathrm{I}^6 - 6\,\mathrm{I}^4\,\mathrm{II} + 6\,\mathrm{I}^3\,\mathrm{III} + 9\,\mathrm{I}^2\,\mathrm{II}^2 - 12\,\mathrm{I\,II\,III} - 2\,\mathrm{II}^3 + 3\,\mathrm{III}^2.
\end{aligned}
$$

(iv) Expand Δ^2, group its terms into elementary polynomials and use the results from (iii) to get the final expression:

$$
\begin{aligned}
\Delta^2 &= -6A + 2B - 2C - 2D + E\\
&= 18\,\mathrm{I\,II\,III} + \mathrm{I}^2\,\mathrm{II}^2 - 4\,\mathrm{I}^3\,\mathrm{III} - 4\,\mathrm{II}^3 - 27\,\mathrm{III}^2.
\end{aligned}
$$